

# THE CURVATURE OF ORBIT SPACES

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ABSTRACT. We investigate orbit spaces of isometric actions on unit spheres and find a universal upper bound for the infimum of their curvatures.

## 1. INTRODUCTION

Let  $G$  be a compact Lie group acting by isometries on the unit sphere  $S^n$ . The space of orbits  $X = S^n/G$  is an Alexandrov space of curvature at least 1 and diameter at most  $\pi$  with respect to the natural quotient metric. The following question of K. Grove has been investigated in [McG93, Gre00, MS05, DGMS09] and remains widely open in general:

*How small can the diameter of the orbit space  $X$  be?*

The group  $G$  can act transitively on  $S^n$ , in which case  $X$  is a point and has diameter zero. Also, the quotient spaces  $S^1/G$  for large cyclic groups  $G$  have arbitrary small nonzero diameters. On the other hand, it has been shown in [Gre00] that for any fixed  $n \geq 2$  there is a gap theorem: for some positive  $\epsilon(n)$ , the diameter of any  $X = S^n/G$  is at least  $\epsilon(n)$  if  $X$  is not a point. It seems plausible that a universal, dimension-independent bound  $\epsilon$  should exist, and the analysis of special classes of actions suggests that such an  $\epsilon$  might be rather large [McG93, Gre00]. The case  $X$  is 1-dimensional is well understood [DGMS09, Theorem B]. In this paper, we consider the case  $\dim(X) \geq 2$  and provide an answer to the following closely related question:

*How curved can the orbit space  $X$  be?*

Denote by  $\kappa_X$  the largest number  $\kappa$  such that the orbit space  $X$  is an Alexandrov space of curvature  $\geq \kappa$ . Note that  $\kappa_X$  equals the infimum of the Riemannian sectional curvatures in the regular part of  $X$ , see Subsection 2.1. Due to the theorem of Bonnet-Myers,  $\kappa_X$  provides an *upper* bound for the diameter, namely  $\text{diam}(X) \leq \pi/\sqrt{\kappa_X}$ . Therefore the existence of a uniform upper bound for  $\kappa_X$  is necessary for the existence of a uniform lower bound for the diameter. The main result of this paper confirms this necessary condition and shows that  $\kappa_X \leq 4$ :

**Theorem.** *Let  $X = S^n/G$  be the orbit space of an isometric action of a compact Lie group  $G$  on the unit sphere  $S^n$  and assume that  $\dim(X) \geq 2$ . If  $X$  is an Alexandrov space of curvature  $\geq \kappa$ , then  $\kappa \leq 4$ .*

This result is sharp as the Hopf action of the circle on  $S^3$  shows. Moreover, there are families of actions for which  $X$  an orbifold of constant curvature 4, see [GL15a] for their classification. On the other hand, one can see that for most

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actions the infimum of the sectional curvatures  $\kappa_X$  of the orbit space  $X$  is equal to 1. It seems to be possible, but much more technical, to determine all actions for which the corresponding number  $\kappa_X$  is larger than 1. We hope to address this classification problem in a forthcoming work. To put Theorem 1 in context, recall that the *supremum* of sectional curvatures in the regular part of  $X$  can be infinite, and it is finite precisely in case  $X$  is a Riemannian orbifold [LT10]; such actions are classified in [GL15a]. We mention that the size and general structure of orbit spaces of unit spheres as in Theorem 1 may have applications to isometric actions on general Riemannian manifolds, cf. [GS97, GW14] and the references therein.

The proof of Theorem 1 uses few simple ideas. Strata in the orbit space are locally convex, thus inheriting the lower curvature bound from their ambient space. On the other hand, any stratum is contained in the principal stratum of the orbit space of another isometric action on a unit sphere, as has been observed in [GL15a, §5.1]. This allows for an inductive approach to the problem and reduces the question to the case where no singular strata of dimension larger than 2 are present. But the absence of such strata implies that the rank of the original group is at most 3. The remaining cases are excluded by an index comparison argument and, in final instance, by the classification of irreducible representations of compact simple Lie groups.

It is an interesting question if our theorem has a geometric explanation not relying on the classification of representations, and if it can be extended to the case of singular Riemannian foliations on the unit sphere. Recently, a large family of singular Riemannian foliations has been constructed in [Rad14], most of which are inhomogeneous (see also [GR15]). All quotient spaces  $X$  arising from these foliations are Alexandrov spaces with curvature  $\geq 4$ , but always have some tangent planes with curvature equal to 4.

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## 2. PRELIMINARIES

**2.1. Orbit spaces and strata.** We recall some basic results about orbit spaces and refer, for instance, to [GL15a, §2] for more details. An isometric action of a compact Lie group  $G$  on a unit sphere is the restriction of an orthogonal representation  $\rho$  of  $G$  on an Euclidean space  $V$ . If  $S(V)$  denotes the unit sphere of  $V$  and  $X = S(V)/G$ , then the cohomogeneity  $c(\rho)$  of  $\rho$  satisfies  $c(\rho) = \dim(X) + 1$ . Let  $\kappa_X = \kappa_\rho$  be defined as in the introduction.

The orbit space  $X$  is an Alexandrov space stratified by smooth Riemannian manifolds, namely, the sets of orbits with the same isotropy groups up to conjugation. Any such stratum  $Y$  is a locally convex subset of  $X$ , so that the infimum of the sectional curvatures along  $Y$  is also bounded below by  $\kappa_X$ . There is exactly one maximal stratum, the set of principal orbits  $X_{\text{reg}}$  of  $X$ , corresponding to the unique minimal conjugacy class of isotropy groups. The corresponding isotropy groups are called the principal isotropy groups. The set of regular points  $X_{\text{reg}}$  is open, dense and convex in  $X$ . The restriction of the projection to the regular set  $S_{\text{reg}}^n \rightarrow X_{\text{reg}}$  is a Riemannian submersion. Moreover, the orbit space  $X$  is the completion of the convex open submanifold  $X_{\text{reg}}$ . Hence Toponogov's globalization theorem, for instance the version in [Pet12], shows that  $\kappa_X$  is equal to the infimum of sectional curvatures of  $X_{\text{reg}}$ .

**2.2. Strata and rank.** Denote the rank of the group  $G$  by  $k$ . Then there exists a point  $p \in S(V)$  such that the isotropy group  $G_p$  has rank at least  $k - 1$  cf. [Wil03, Lemma 6.1]. We infer:

**Lemma 2.1.** *Let a group  $G$  act by isometries on  $S(V)$ . If  $G$  has rank  $k$ , then the minimal dimension  $\ell$  of a  $G$ -orbit is at most  $\dim(G) - k + 1$ .*

We will need another simple observation:

**Lemma 2.2.** *Let a group  $G$  of rank  $k$  act by isometries on  $S(V)$ . If the  $G$ -action has trivial principal isotropy groups, then  $X = S(V)/G$  contains a non-maximal stratum of dimension at least  $k - 2$ .*

*Proof.* We proceed by induction on  $k$ . In the initial case  $k = 2$ , there exists a point with non-trivial isotropy group, so the quotient  $X$  contains non-regular points, and therefore at least one non-maximal stratum (of dimension at least 0). In general, we first find a point  $p \in S(V)$  such that the isotropy group  $G_p$  has rank at least  $k - 1$ . The slice representation of  $G_p$  on the normal space  $\mathcal{H}_p$  has again trivial principal isotropy groups, so the inductive assumption yields that the quotient of the unit sphere  $S(\mathcal{H}_p)$  by  $G_p$  has a non-regular stratum of dimension at least  $k - 3$ . The corresponding stratum in  $\mathcal{H}_p/G_p$ , and thus also in a neighborhood of  $x = G \cdot p$  in  $X$ , has dimension at least  $k - 3 + 1 = k - 2$ , which completes the induction step.  $\square$

**2.3. Enlarging group actions and polar representations.** Let  $\tau : H \rightarrow \mathcal{O}(V)$  be a representation of a compact Lie group. Consider a closed subgroup  $G$  of  $H$  and the representation  $\rho : G \rightarrow \mathcal{O}(V)$  obtained by restriction. The canonical projection  $S(V)/G \rightarrow S(V)/H$  restricts to a Riemannian submersion on an open and dense set, hence this map does not decrease the sectional curvatures by the formula of O'Neill [GW09, Corollary 1.5.1]. This shows:

**Proposition 2.1.** *Suppose an orthogonal representation  $\rho : G \rightarrow \mathcal{O}(V)$  is the restriction of another representation  $\tau : H \rightarrow \mathcal{O}(V)$ , where  $G$  is a closed subgroup of  $H$ . If  $c(\tau) \geq 3$  then  $\kappa_\rho \leq \kappa_\tau$ .*

Recall that polar representations  $\rho : G \rightarrow \mathcal{O}(V)$  are exactly those whose induced action on  $S(V)$  has orbit space of constant curvature 1 [GL15b, Introd.]. In particular, all polar representations  $\tau$  with  $c(\tau) \geq 3$  satisfy  $\kappa_\tau = 1$ . We now have:

**Corollary 2.1.** *Let  $\rho_i : G_i \rightarrow \mathcal{O}(V_i)$  for  $i = 1, 2$  be  $\mathbb{F}$ -linear representations, where  $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ . If  $\dim_{\mathbb{F}}(V_i) \geq 3$  for  $i = 1, 2$ , then the tensor product representation  $\rho$  of  $G = G_1 \times G_2$  on  $V = V_1 \otimes_{\mathbb{F}} V_2$  satisfies  $\kappa_\rho = 1$ .*

*Proof.* It follows directly from Proposition 2.1 by taking  $H = \mathcal{O}(V_1) \otimes \mathcal{O}(V_2)$ ,  $H = \mathcal{U}(V_1) \otimes \mathcal{U}(V_2)$ ,  $H = \mathcal{Sp}(V_1) \otimes \mathcal{Sp}(V_2)$  if  $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , respectively. Indeed, the representation  $\tau$  of  $H$  on  $V$  is polar and enlarges  $\rho$ . Moreover, the cohomogeneity  $c(\tau)$  is the minimum of the two numbers  $\dim_{\mathbb{F}}(V_i) \geq 3$ .  $\square$

**2.4. Reductions and strata as subquotients.** Let a compact Lie group  $G$  act by isometries on a unit sphere  $S(V)$ . Fix an arbitrary point  $p \in S(V)$ , denote its isotropy group by  $G_p$ , consider its orbit  $x = G \cdot p \in X = S(V)/G$ , and denote by  $Y$  the stratum of  $X$  which contains  $x$ . The set of all points in  $V$  fixed by  $G_p$  is a subspace  $W$  on which the normalizer  $N$  of  $G_p$  in  $G$  acts isometrically. There is a canonical map  $S(W)/N \rightarrow X = S(V)/G$ , which is length-preserving and a local

isometry on an open dense subset of  $S(W)/N$  [GL15a, Lemma 5.1]. Moreover,  $Y$  is dense in the image of this map. We deduce that the infimum of the curvatures on  $S(W)/N$  is also bounded from below by  $\kappa_X$ . Finally, note that  $G_p$  acts trivially on  $W$  and that the induced action of  $H = N/G_p$  on  $S(W)$  has trivial principal isotropy groups. This shows:

**Proposition 2.2.** *Let  $\rho : G \rightarrow \mathrm{O}(V)$  be a representation with  $c(\rho) \geq 3$ . Let  $X = S(V)/G$  and assume that  $X$  has a singular stratum  $Y$  of dimension  $d \geq 2$ . Then there exists a representation  $\tau : H \rightarrow \mathrm{O}(W)$  with trivial principal isotropy groups such that  $S(W)/H$  has dimension  $d$  and  $\kappa_\rho \leq \kappa_\tau$ .*

**2.5. An index estimate.** Let  $G$  act on  $S(V)$  as above and consider the restriction  $\pi$  of the projection  $S(V) \rightarrow X$  to the regular part. Applying O'Neill's formula for the curvature of the Riemannian submersion  $\pi$ , we see that planes with curvature 1 exist in  $X = S(V)/G$  if and only if O'Neill's tensor of  $\pi$  vanishes at some pair of linearly independent, horizontal vectors. Therefore,  $\kappa_\rho > 1$  directly implies that the dimension of a regular orbit  $G \cdot p$  is at least  $\dim(X) - 1$ . On the other hand, using index estimates one can say slightly more.

Indeed let  $F = G \cdot p$  be a principal orbit and let  $\gamma : [0, \pi] \rightarrow S^n$  be a unit speed  $F$ -geodesic, namely, a geodesic starting perpendicularly to  $F$ . The index of  $\gamma$  as  $F$ -geodesic (which equals the sum of focal multiplicities of  $F$  along  $\gamma$ ) is equal to  $\dim(F)$ . On the other hand, the index of  $\gamma$  is also obtained as the sum of the vertical index  $\mathrm{ind}_{\mathrm{vert}}(\gamma)$  and the horizontal index  $\mathrm{ind}_{\mathrm{hor}}(\gamma)$ , see [LT10, Lemma 5.1] and [Lyt09, §3]. The vertical index counts the intersections of  $\gamma$  with singular orbits:

$$\mathrm{ind}_{\mathrm{vert}}(\gamma) = \sum_{t \in (0, \pi)} (\dim(F) - \dim(G \cdot \gamma(t))).$$

Moreover, the horizontal index is the index of the transversal Jacobi equation defined by Wilking in [Wil07]. This Jacobi equation has the form  $J'' + R_t(J) = 0$  for a time-dependent symmetric operator  $R_t : U \rightarrow U$  on a Euclidean vector space  $U$  of dimension  $\dim(X) - 1$ . Around a regular point  $\gamma(t)$ , the Jacobi equation  $J'' + R_t(J) = 0$  is just the Jacobi equation in the Riemannian manifold  $X_{\mathrm{reg}}$  along the projected geodesic  $\pi \circ \gamma$ . Therefore, if all sectional curvatures at regular points of  $X$  along  $\gamma$  are at least  $\kappa$ , then  $R_t \geq \kappa \cdot \mathrm{Id}$ . Thus the standard index comparison [Kli95, Lemma 2.6.1] implies that  $\mathrm{ind}_{\mathrm{hor}}(\gamma) \geq \dim(X) - 1$  in case  $\kappa_\rho > 1$ , and  $\mathrm{ind}_{\mathrm{hor}}(\gamma) \geq 2(\dim(X) - 1)$  in case  $\kappa_\rho > 4$ .

Now we can easily deduce:

**Proposition 2.3.** *Let  $\rho : G \rightarrow \mathrm{O}(V)$  be as above. Denote by  $\ell$  the smallest dimension of a  $G$ -orbit in  $S(V)$ , and by  $m \geq 2$  be the dimension of the orbit space  $X = S(V)/G$ . Then:*

- (1) *if  $\kappa > 1$ , then  $\ell \geq m - 1$ ;*
- (2) *if  $\kappa > 4$ , then  $\ell \geq 2(m - 1)$ .*

*Proof.* Let  $L$  be an orbit of smallest dimension  $\ell$ . Take a regular orbit  $F$  and a horizontal unit speed geodesic  $\gamma : [0, \pi] \rightarrow S(V)$  starting in  $F$  and going through  $L$ . The index of  $\gamma$  is  $\dim(F)$ . On the other hand, the vertical index is at least  $\dim(F) - \ell$ . Since the index of  $\gamma$  is the sum of the vertical and the horizontal indices, we see that  $\ell$  cannot be smaller than the horizontal index of  $\gamma$ . Thus the result follows from the index estimates above.  $\square$

### 3. MAIN RESULT

**3.1. Formulation.** In this section, we prove Theorem 1. Suppose to the contrary that there exists a representation  $\rho : G \rightarrow \mathbf{O}(V)$  a compact Lie group  $G$  such that  $X = S(V)/G$  has dimension  $m \geq 2$  and satisfies  $\kappa_\rho > 4$ . We may assume that  $m$  is minimal among all such examples. Namely, for any representation  $\tau : H \rightarrow \mathbf{O}(W)$  of a compact Lie group  $H$  such that the dimension  $m'$  of the orbit space  $Y = S(W)/H$  satisfies  $m > m' \geq 2$ , we have  $\kappa_\tau \leq 4$ . We assume further that, for this  $m$ ,  $g := \dim(G)$  is minimal among all such examples, and we fix the representation  $\rho$  throughout the proof.

**3.2. Principal isotropy and identity component.** By the assumption on the minimality of  $g$ , the representation of  $\rho$  is *reduced* in the sense of [GL14, §1.2]: for any other representation  $\tau : H \rightarrow \mathbf{O}(W)$  such that  $S(W)/H$  is isometric to  $X$ , we have  $\dim(H) \geq g$ . In particular, this implies that the action of  $G$  on  $S(V)$  has trivial principal isotropy groups.

Let  $\rho_0$  be the restriction of  $\rho$  to the identity component  $G^0$  of  $G$ . Then the projection  $X_0 = S(V)/G^0 \rightarrow X = S(V)/G$  is a Riemannian covering over the set of regular points of  $X$ . We deduce  $\kappa_\rho = \kappa_{\rho_0}$ . Hence, we may replace  $G$  by  $G^0$  and assume from now on that  $G$  is connected.

**3.3. Strata and rank.** The minimality assumption on  $m$  together with Proposition 2.2 show that  $X$  does not contain singular strata of dimensions larger than 1. From Lemma 2.2 we deduce that the rank  $k$  of  $G$  is at most 3.

**3.4. Irreducibility.** Our assumption yields  $\text{diam}(X) \leq \pi/\sqrt{\kappa_X} < \pi/2$ . This implies that the representation  $\rho : G \rightarrow \mathbf{O}(V)$  is irreducible [GL14, §5].

**3.5. Basic identity and inequality.** Now  $\rho$  is an irreducible representation of a connected group  $G$  of dimension  $g$  and rank  $1 \leq k \leq 3$ . The representation is also reduced, so the principal isotropy groups are trivial and the dimension  $n$  of  $S(V)$  satisfies

$$g + m = n.$$

From Lemma 2.1 and Proposition 2.3 we deduce

$$g - k \geq 2m - 3.$$

In particular, we have  $n \leq \frac{3}{2}g + 1$ .

**3.6. The case  $m = 2$ .** Due to [Str94], in this case the only reduced representations  $\rho : G \rightarrow \mathbf{O}(V)$  of a connected non-trivial group  $G$  are representations of the circle group  $\mathbf{U}(1)$  of  $\mathbb{R}^4$ . But such representations are reducible.

**3.7. The case  $m = 3$ .** Due to the classification of irreducible representations of cohomogeneity 4 of connected compact Lie groups [GL14, Theorem 1.8], in this case the only reduced representations are given by the actions of  $\mathbf{SO}(3)$  on  $\mathbb{R}^7$  and by  $\mathbf{U}(2)$  on  $\mathbb{R}^8$ . Both cases contradict the inequality  $g - k \geq 2m - 3 = 3$ .

**3.8. The case  $m = 4$ .** Due to the classification of irreducible representations of cohomogeneity 5 of connected compact Lie groups [GL14, Theorem 1.8], in this case the only reduced representations are given by the action of  $\mathbf{SU}(2)$  on  $\mathbb{R}^8$  and by the action of  $\mathbf{SO}(3) \times \mathbf{U}(2)$  on  $\mathbb{R}^{12} = \mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4$ . Both cases contradict the inequality  $g - k \geq 2m - 3 = 5$ .

**3.9. Consequences.** Henceforth we may assume that  $m \geq 5$ . Therefore  $g \geq 2m - 3 + k \geq 7 + k$ . This immediately excludes the possibility that  $G$  have rank 1 or that  $G$  be covered by a product of (two or three) groups of rank 1. Moreover,  $G$  cannot be covered by  $\mathrm{SU}(3)$  or  $\mathrm{U}(1) \times \mathrm{SU}(3)$ .

**3.10. Type of representation.** We claim that the centralizer  $N$  of  $\rho(G)$  in  $\mathrm{O}(V)$  has  $\rho(G)$  as its identity component. Otherwise, we find a subgroup  $H$  of  $\mathrm{O}(V)$  containing  $\rho(G)$  with one dimension more. The inclusion  $\tau : H \rightarrow \mathrm{O}(V)$  is an enlargement of  $\rho$ . The quotient space  $Y = S(V)/H$  has dimension at least  $m - 1 \geq 4$ , and  $\kappa_\rho \leq \kappa_\tau$  owing to Proposition 2.1. By our minimality assumption on  $m$ , we must have  $\dim(Y) = \dim(X)$ . But then  $\tau$  and  $\rho$  have the same orbits. This implies that  $\tau$  has non-trivial principal isotropy groups, so it cannot be reduced. It follows that neither  $\rho$  is reduced, in contradiction to our assumption.

We deduce that  $\rho$  cannot be of quaternionic type, and if  $\rho$  is of complex type then  $G$  is covered by  $\mathrm{U}(1) \times G'$  for some connected compact Lie group  $G'$ .

**3.11. Representations of complex type.** Assume that  $\rho$  is of complex type, namely,  $V$  admits an invariant complex structure; in particular,  $\dim(V)$  is even. It follows from the preceding two subsections that  $G$  must be covered by  $\mathrm{U}(1) \times G'$ , where  $G'$  a connected compact simple Lie group of rank 2 and  $G' \neq \mathrm{SU}(3)$ . Hence  $G' = \mathrm{Sp}(2)$  or  $\mathrm{G}_2$ .

If  $G' = \mathrm{Sp}(2)$ , we have  $g = 11$  and  $11 \geq 2m$ , which gives  $m = 5$ . Then the dimension  $n + 1$  of  $V$  is 17, which is odd and thus impossible.

If  $G' = \mathrm{G}_2$ , we obtain  $g = 15$ , hence  $15 \geq 2m$ . Since the dimension of  $V$  is even, we deduce that  $m$  must be even as well, hence  $m = 6$ . Now  $V$  has dimension  $g + m + 1 = 22$ . But there are no irreducible representations of  $\mathrm{G}_2$  on  $\mathbb{C}^{11}$  (cf. appendix in [Kol02]).

There remains only the case  $\rho$  is of real type.

**3.12. Non-simple groups.** If  $G$  is not simple then, up to a finite covering, it must have the form  $G = G_1 \times G_2$  where  $\mathrm{rk}(G_1) = 2$  and  $\mathrm{rk}(G_2) = 1$  and where  $G_1$  is simple. Now  $G_1$  is one of  $\mathrm{SU}(3)$ ,  $\mathrm{Sp}(2)$  or  $\mathrm{G}_2$ . Since  $\rho$  is irreducible and there is no  $G$ -invariant complex structure on  $V$ ,  $G_2$  cannot be  $\mathrm{U}(1)$ , so it is covered by  $\mathrm{SU}(2)$ . The representation  $\rho$  is a tensor product  $\rho = \rho_1 \otimes_{\mathbb{F}} \rho_2$  where the  $\rho_i : G_i \rightarrow \mathrm{O}(V_i)$  are irreducible  $\mathbb{F}$ -linear representations and  $\mathbb{F} = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Since the representation  $\rho$  is of real type, we cannot have  $\mathbb{F} = \mathbb{C}$ . From Corollary 2.1, we see that the  $\mathbb{F}$ -dimensions of  $V_i$  cannot be both larger than 3. We deduce that  $\mathbb{F}$  cannot be  $\mathbb{R}$  either, thus  $\mathbb{F} = \mathbb{H}$ .

Note that  $\mathrm{SU}(3)$  and  $\mathrm{G}_2$  do not have irreducible representations of quaternionic type. We are left with the case  $G_1 = \mathrm{Sp}(2)$ ,  $G_2 = \mathrm{SU}(2)$ . Then  $g = 13$  and  $13 \geq 2m$ . Since the dimension  $g + m + 1$  of  $V$  must be divisible by 4, we get  $m = 6$  and  $\dim(V) = 20$ . Then  $20 = 4rs$ , where  $r$  and  $s$  are the dimensions over  $\mathbb{H}$  of  $V_1$  and  $V_2$ , respectively. We deduce  $r = 5$  and  $s = 1$ . However, there does not exist irreducible quaternionic representation of  $\mathrm{Sp}(2)$  on  $\mathbb{H}$  or  $\mathbb{H}^5$  (cf. again the appendix in [Kol02]).

**3.13. Kollross' table and the case of simple groups.** We are left with the case  $G$  is a simple group of rank  $2 \leq k \leq 3$ . The dimension  $n + 1$  of  $V$ , thus the degree of the corresponding complexified representation satisfies  $n + 1 \leq \frac{3}{2}g + 2 < 2g + 2$ . Thus we may apply Lemma 2.6 from [Kol02] to deduce that  $\rho$  is one of the



representations listed in the tableau therein. Only four of those representations are representations of groups of rank  $2 \leq k \leq 3$ , namely two representations of  $G_2$  and two representations of  $\text{Spin}(7)$ . Two of these representations satisfy  $g \geq \dim(V)$  (they also have cohomogeneity one), which is impossible under our assumptions. For the two remaining representations the condition  $n \leq \frac{3}{2}g + 1$  is violated. This finishes the proof of Theorem 1.

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